8.3 Trigonometric integrals

Goal: Evaluate: $\int \sin^m x \cos^n x \, dx$

m, n are whole numbers

(1) Assuming at least one of the constants m, n is odd

Tool: $\sin^2 x = 1 - \cos^2 x$ or $\cos^2 x = 1 - \sin^2 x$

Eg: $\int \sin^3 x \, dx$ $(m = 3)$ odd

$= \int \sin x \cdot \sin^2 x \, dx$

$= \int \sin x (1 - \cos^2 x) \, dx$

$= \int \sin x \, dx - \int \sin x \cos^2 x \, dx$

$= -\cos x - \int (-u^2) \, du$ let $u = \cos x$, $du = -\sin x \, dx$

$= -\cos x + \int u^2 \, du = -\cos x + \frac{1}{3} u^3 + C$

$\boxed{= -\cos x + \frac{1}{3} \cos^3 x + C}$

Eg: $\int \sin^2 x \cos^5 x \, dx$

Tool: lowering power formula for cosine:

$\checkmark \frac{1+\cos 2x}{2} = \cos^2 x$

$\frac{1-\cos 2x}{2} = \sin^2 x$

$$= \int \sin^2 x \left[\cos^2 x\right]^2 \cos x \, dx$$

$$= \int \sin^2 x \left[1 - \sin^2 x\right]^2 \cos x \, dx$$

$$= \int \sin^2 x \left(1 - 2\sin^2 x + \sin^4 x\right) \cos x \, dx$$

$$= \int \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) \underline{\cos x \, dx}$$

Let $u = \sin x$
$du = \cos x \, dx$

$$= \int \left(u^2 - 2u^4 + u^6\right) du$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \boxed{\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}$$

$$\int \sin^2 x \, dx \qquad (m = 2 \quad \text{even})$$

$$= \int \left[\frac{1 - \cos 2x}{2} \right] dx$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \int 1 \, dx - \frac{1}{2} \int \cos 2x \, dx$$

$u = 2x \qquad du = 2\,dx$

$\frac{1}{2} du = dx$

$$= \frac{1}{2} x - \frac{1}{2} \int \cos u \left(\frac{1}{2} du \right)$$

$$= \frac{1}{2} x - \frac{1}{4} \sin u + C = \boxed{\frac{1}{2} x - \frac{1}{4} \sin 2x + C}$$

8.4 Trigonometric substitution:

Goal: integrate expressions that involve: $\sqrt{a^2-x^2}$; $\sqrt{x^2-a^2}$; $\sqrt{x^2+a^2}$ $a>0$

① $\sqrt{a^2-x^2}$ substitution: $x = a\sin\theta$

$-a < x < a$

Eg: $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

Solution: Let $x = 2\sin\theta \longrightarrow dx = 2\cos\theta\, d\theta$

$$\Longrightarrow \int \frac{1}{\sqrt{4-[\underbrace{2\sin\theta}_{x}]^2}}[\underbrace{2\cos\theta\, d\theta}_{dx}]$$

$$\Longrightarrow \int \frac{1}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\, d\theta = \int \frac{1}{\cancel{2}\sqrt{1-\sin^2\theta}}\,\cancel{2}\cos\theta\, d\theta$$

$$\Longrightarrow \int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}} \longrightarrow \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta$$

$$\longrightarrow \theta + C$$

$x = 2\sin\theta$

$$\longrightarrow \boxed{\sin^{-1}\left(\frac{x}{2}\right) + C}$$

$\rightarrow \frac{x}{2} = \sin\theta$

$\rightarrow \sin^{-1}\left(\frac{x}{2}\right) = \theta$

② $\sqrt{x^2+a^2}$ Let $x = a\tan\theta$, $a>0$

Eg: Evaluate

$$\int\sqrt{3+x^2}\,dx \qquad a=\sqrt{3} \ \ (\text{since } a^2=3)$$

Let $x=\sqrt{3}\tan\theta$

$dx=\sqrt{3}\sec^2\theta\,d\theta$

$$\Longrightarrow \int\sqrt{(\sqrt{3})^2+\underbrace{(\sqrt{3})^2\tan^2\theta}_{x}}\,\left[\sqrt{3}\sec^2\theta\,d\theta\right]$$

$\sec^2\theta = 1+\tan^2\theta$

$$\Longrightarrow \int\sqrt{3}\sqrt{1+\tan^2\theta}\,[\sqrt{3}]\sec^2\theta\,d\theta$$

$$\Longrightarrow 3\int\sqrt{\sec^2\theta}\,\sec^2\theta\,d\theta = 3\int\sec^3\theta\,d\theta$$

$$\Longrightarrow 3\int\sec\theta\cdot\sec^2\theta\,d\theta$$

$$\Longrightarrow 3\int\sec\theta\,(1+\tan^2\theta)\,d\theta$$

$$\Longrightarrow 3\int\sec\theta\,d\theta + 3\int\sec\theta\tan^2\theta\,d\theta$$

So $3\int\sec^3\theta\,d\theta = 3\ln|\sec\theta+\tan\theta| + 3\int\underbrace{\sec\theta\tan\theta}_{dv}\cdot\underbrace{\tan\theta}_{u}\,d\theta$

* by parts

* $3\int\sec\theta\tan\theta\tan\theta\,d\theta$

$u=\tan\theta \to du=\sec^2\theta\,d\theta$

$dv=\sec\theta\tan\theta\,d\theta \to v=\int\sec\theta\tan\theta\,d\theta = \sec\theta$

$uv-\int v\,du = \tan\theta\sec\theta - \int\sec^3\theta\,d\theta$

So $3\int\sec\theta\tan^2\theta\,d\theta = 3\tan\theta\sec\theta - 3\int\sec^3\theta\,d\theta$

Now $3\int\sec^3\theta\,d\theta = 3\ln|\sec\theta+\tan\theta| + 3\tan\theta\sec\theta - 3\int\sec^3\theta\,d\theta$

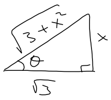
add $3\int\sec^3\theta\,d\theta$ on both sides yields

$$6\int\sec^3\theta\,d\theta = 3\ln|\sec\theta+\tan\theta| + 3\tan\theta\sec\theta$$

Therefore (divide both sides by 3) we have

$$\int\sqrt{3+x^2}\,dx = 3\int\sec^3\theta\,d\theta = \ln|\sec\theta+\tan\theta| + \tan\theta\sec\theta + C$$

Since $x=\sqrt{3}\tan\theta$, $\longrightarrow \tan\theta = \frac{x}{\sqrt{3}} \to$



Now, $\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{3+x^2}}{\sqrt{3}} = \frac{\sqrt{9+3x^2}}{3}$

and $\tan\theta = \frac{opp}{adj} = \frac{x}{\sqrt{3}} = \frac{x\sqrt{3}}{3}$

Therefore,

$$\int\sqrt{3+x^2}\,dx = \ln\left|\frac{\sqrt{9+3x^2}}{3}+\frac{x\sqrt{3}}{3}\right| + \frac{x\sqrt{3}}{3}\cdot\frac{\sqrt{9+3x^2}}{3} + C$$

$$= \ln\left|\frac{\sqrt{9+3x^2}+x\sqrt{3}}{3}\right| + \frac{x\sqrt{3+x^2}}{3} + C$$

Warning: Do not attempt this during a commercial break!!!

③ $\sqrt{x^2-a^2}$ let $x = a\sec\theta$

Eg: $\displaystyle\int \frac{1}{x^2\sqrt{4x^2-36}}\,dx$ $\qquad \displaystyle\frac{1}{2}\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$

Note: $\sqrt{4x^2-36} = 2\sqrt{x^2-9}$

So we let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$

$$\Longrightarrow \frac{1}{2}\int \frac{1}{(3\sec\theta)^2\sqrt{(3\sec\theta)^2-9}}\,(3\sec\theta\tan\theta\,d\theta)$$

$$\Longrightarrow \frac{1}{18}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}}\,d\theta$$

$\tan^2\theta+1 = \sec^2\theta$

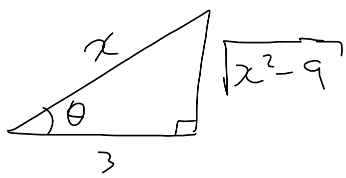
$\tan^2\theta = \sec^2\theta-1$

$$\Longrightarrow \frac{1}{18}\int \frac{\tan\theta}{\sec\theta\tan\theta}\,d\theta = \frac{1}{18}\int \frac{1}{\sec\theta}\,d\theta$$

$$\Longrightarrow \frac{1}{18}\int \cos\theta\,d\theta = \frac{1}{18}\sin\theta + C$$

Note: $x = 3\sec\theta$

$\dfrac{x}{3} = \sec\theta = \dfrac{hyp}{adj}$



So, $\sin\theta = \dfrac{opp}{hyp} = \dfrac{\sqrt{x^2-9}}{x}$

$$\boxed{\frac{1}{18}\left[\frac{\sqrt{x^2-9}}{x}\right] + C}$$